Bulgaria International Mathematics Competition 2023 (Virtual)

Bulgaria, 1st to 7th July 2023

Invitational World Youth Mathematics Intercity Competition

Individual Contest

Time limit: 120 minutes

Information:

- You are allowed 120 minutes for this paper, consisting of 12 questions in Section A to which only numerical answers are required, and 3 questions in Section B to which full solutions are required.
- Each question in Section A is worth 5 points. No partial credit is given. There are no penalties for incorrect answers, but you must not give more than the number of answers being asked for. For questions asking for several answers, full credit will only be given if all correct answers are found. Each question in Section B is worth 20 points. Partial credit may be awarded.
- Diagrams shown may not be drawn to scale.

Instructions:

- Key in your name, your contestant ID and your team's name before going to next page.
- You may not use instruments such as protractors, calculators and electronic devices.
- For Section A, Enter your answers in the column provided after each question.
  For all answers, **no need to key in their units**. The format as following:
  1. For decimal *a*.*bc*, where *a*,*b* and *c* are digits, key in *a*.*bc*.
  2. For fraction $\frac{a}{b}$, where *a* and *b* are coprime, key in *a*/*b* (For example, if your answer is $3\frac{2}{5}$, please key in **17/5**).
  3. For ratio $a:b$, key in *a*:*b* or *a*;*b* (no need space after ":" or ";").
  4. For number pair (*a*, *b*, *c*, …), key in *a*,*b*,*c*,….(no need space after ",").
  5. If the solution is $a+b\times\sqrt{c}$, please key in *a*,*b*,*c* (no need space after ",".
     For example, **if your answer is** $3+\sqrt{5}$**, please key in 3,1,5**).
- At the end of the contest, you must click "send" for the Section A questions and scan or take a photo of your Section B questions solutions then upload to the given website.

English Version

Team: ____________ ***Name:*** ________________ ***ID.:*** ____________

Bulgaria International Mathematics Competition 2023 (Virtual)

Bulgaria, 1st to 7th July 2023

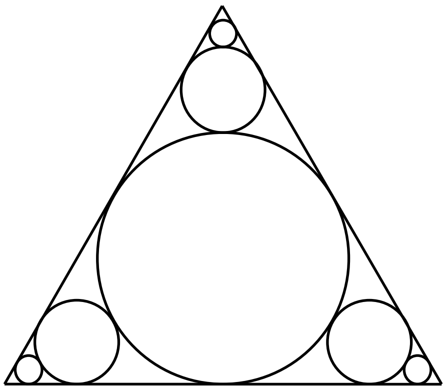
Invitational World Youth Mathematics Intercity Competition

Section A.

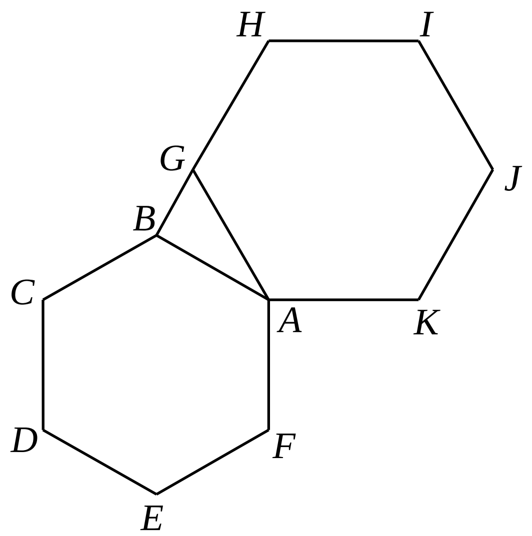
In this section, there are 12 questions. Fill in the correct answer in the space provided at the end of each question. Each correct answer is worth 5 points.

1. Let x be some real number, and $y=\sqrt{3x+4}+\sqrt{4-3x}$.
   What is the minimum possible value of y^2?

2. A circle of radius 1 cm is drawn touching the three edges of an equilateral triangle. Starting from this circle, three infinite sequences of smaller circles are then drawn, one at each corner, such that each circle touches tangentially the previous circle and two of the edges of the triangle, as shown in the diagram below. What is the sum of the circumferences, in cm, of all the circles? (Take $\pi=3.14$)



3. Let $ABCDEF$ and $AGHIJK$ be two regular hexagons that do not overlap, as shown in the diagram below. Assume that $\angle FAK=90°$ and that the areas satisfy $3\times[AGHIJK]=4\times[ABCDEF]$. What is the ratio of $[ABG]:[AGHIJK]$?
   (Note: $[P]$ denotes the area of polygon P.)



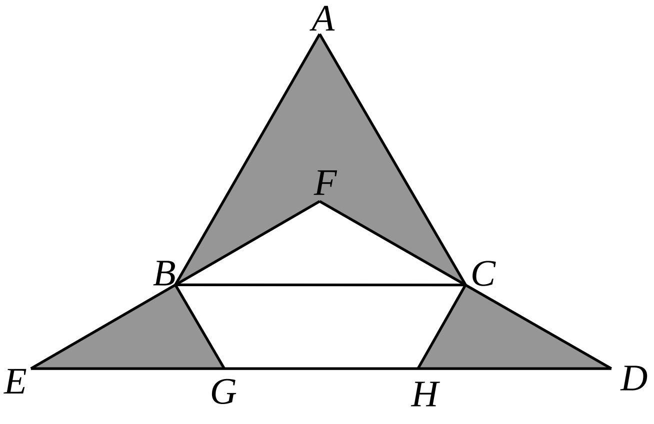
4. The 53-digit number
37,984,318,966,591,152,105,649,545,470,741,788,308,402,068,827,142,719
can be expressed as n^{21}, where n is a positive integer. What is the value of n?

5. Consider the equation $10y^2-9x^{2022}=y^4$, where x and y are integers. If m is the greatest possible value of $x+y$ and n is the number of solutions (x, y), what is $m+n$?

6. Let a, b and c be three non-zero real numbers that satisfy $a+b+c=0$. What is the value of $S=\dfrac{a^4}{a^4-(b^2-c^2)^2}+\dfrac{b^4}{b^4-(c^2-a^2)^2}+\dfrac{c^4}{c^4-(a^2-b^2)^2}$?

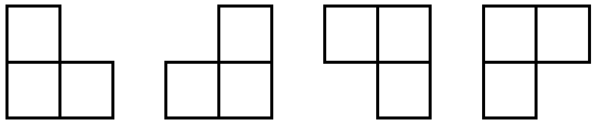
7. If a number can be expressed as 2^a+2^b, where a, b are non-negative integers and $a \neq b$, then this number is called a "*Lucky number*." Suppose all lucky numbers are listed in increasing order. What is the 64th lucky number?

8. Let ABC be an equilateral triangle whose area is 36 cm^2, and EFD be an isosceles triangle with $EF=FD$. The point F is the centre of triangle ABC, and the points B and C are the midpoints of EF and FD, respectively, as shown in the diagram below. If $BG \perp EF$ and $CH \perp DF$, what is the area, in cm^2, of the shaded region?



9. A very secret company develops a hi-tech machine. There are 20 different blueprints needed to construct it. Every employee has access to exactly 5 different blueprints and every combination of 5 different blueprints is accessible by at least one employee. The CEO wants to distribute employees into departments so that no department could make the machine by itself. That is, there should not be any department where every blueprint is accessible by at least one of its members. What is the smallest number of departments the CEO needs to create?

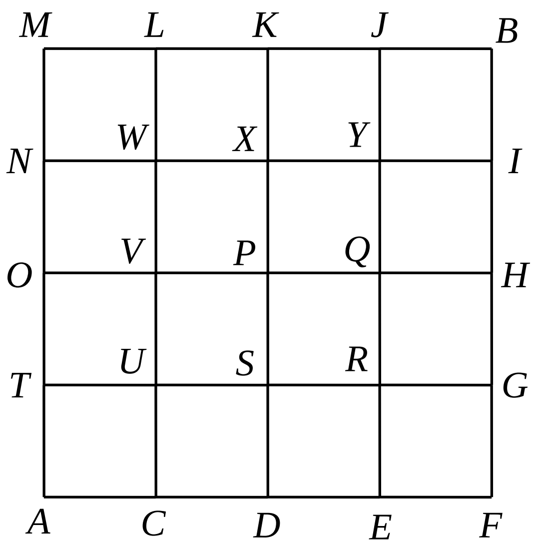
10. A V-Tromino is made up of three 1×1 squares, as diagrams shown below:



All the V-Trominos must be aligned with the board cells. What is the minimum number of the V-Trominos needed to be placed on an 8×8 board so that one cannot fit a 2×2 square in the remaining space?

11. There are 17 empty boxes and an infinite supply of balls. In each step, we choose some number of boxes, then put in each chosen box a different number of balls, where each number must be a power of 2 (including 2 to the power of 0). After *k* steps, it's possible that all boxes have the same non-zero number of balls inside. What is the smallest positive integer *k* to achieve this?

12. The diagram given below shows an underground maze in the form of a 4×4 square. A snake, initially at position *P*, starts crawling along the path *P*-*Q*-*R*-*S* in a loop. A mouse, initially at position *A*, has to reach position *B*, moving across the maze. The mouse can only move vertically up or horizontally right. If the snake and the mouse reach the same position at the same time, the snake will eat the mouse; also if the two cross each other along a path, the snake will eat the mouse (the snake and the mouse move at the same speed). Given that the mouse and the snake start moving at the same time, what is the number of safe paths for the mouse to move from *A* to *B*?
(For example, *A*-*C*-*D*-*E*-*R*-*G*-*H*-*I*-*B* is a safe path for the mouse, since when the mouse reaches *R*, the crawling snake will be at point *P*)



Section B.
Answer the following 3 questions, and show your detailed solution in the space provided after each question. Each question is worth 20 points.

1. What is the number of real number solutions of the equation $x^2-8[x]+7=0$? Note: $[x]$ denotes the largest integer that is not greater than x. For example, $[3.14]=3$ and $[-3.14]=-4$.

Answer: ______________________

2. Consider a regular hexagon. Paul wishes to colour each of the vertices of the hexagon in green, red or blue, in such a way that there are no neighbouring vertices coloured in the same colour. What is the total number of ways Paul can do such a colouring?

Answer: ______________ ways

3. In triangle ABC, M is the midpoint of BC. We draw the circle with centre O that passes through the points A and C and is tangent to the line AM. The extension of BA intersects the circle at D and the extension of CD intersects the extension of MA at P, as shown in the diagram below. Prove that $OP \perp BC$.