India International Mathematics Competition 2024

Lucknow, 26th to 31st July 2024

Invitational World Youth Mathematics Intercity Competition

Individual Contest

Time limit: 120 minutes

Information:

- You are allowed 120 minutes for this paper, consisting of 12 questions in Section A to which only numerical answers are required, and 3 questions in Section B to which full solutions are required.
- Each question in Section A is worth 5 points. No partial credits are given. There are no penalties for incorrect answers, but you must not give more than the number of answers being asked for. For questions asking for several answers, full credit will only be given if all correct answers are found. Each question in Section B is worth 20 points. Partial credits may be awarded.
- Diagrams shown may not be drawn to scale.

Instructions:

- Write down your name, your contestant ID and your team's name on the answer sheet of Section A and on **each page of Section B**.
- For Section A, write your answers in the space provided on the answer sheet.
  For Section B, write down your solutions on spaces provided after individual questions.
- You are allowed to use HB, B, and 2B pencils and ball-point pens with black or blue ink.
- You cannot use instruments such as protractors, calculators and electronic devices.
- At the end of the contest, you must hand in the envelope containing the question sheets, the answer sheet and all scrap paper.

English Version

Section A.

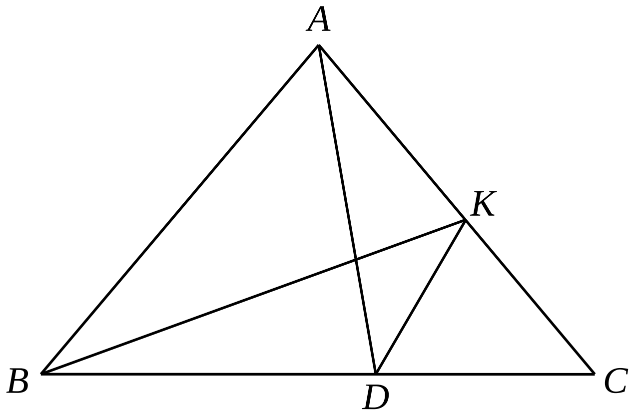
In this section, there are 12 questions. Fill in the correct answer in the space provided in the Answer sheet. Each correct answer is worth 5 points.

1. Let $N=\dfrac{1^4+2022^4+2023^4}{1^2+2022^2+2023^2}$. What is the remainder when N is divided by 2024?

2. Let a and b be positive real numbers such that $\dfrac{a}{a+b^2}+\dfrac{b}{b+a^2}=\dfrac{7}{8}$ and $ab=2$. What is the value of a^9+b^9?

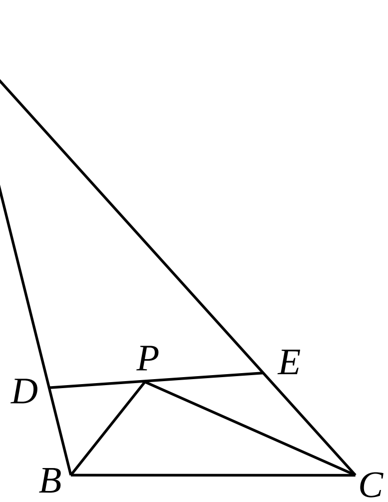
3. Let a and b be positive integers such that $(a+1)(a+2)(a+3)...(a+b)=2527k$, where k is a positive integer. What is the smallest possible value of $a+b$?

4. In the diagram below, ABC is an isosceles triangle with $AB=AC$ and $\angle BAC=80°$. The points D and K are on BC and AC, respectively, such that $\angle CAD=\angle ABK=30°$. What is the measure, in degrees, of $\angle BKD$?



5. Let α and β be the two roots of the equation $x^2-x-1=0$. What is the value of $\alpha^9+13\alpha^8\beta^9+\beta^8$?

6. In the diagram below, ABC is a triangle, where the angle bisectors of $\angle ABC$ and $\angle ACB$ intersect at point P. Let points D and E lie on sides AB and AC, respectively, such that DE passes through point P and $\angle AED=\angle ABP$. If the areas of triangles BDP, CEP and BPC are equal to 18 cm^2, 36 cm^2 and 57 cm^2, respectively, then what is the area, in cm^2, of ABC?

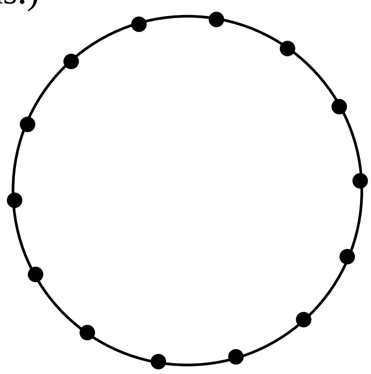


7. Use three different digits A, B and C exactly once to create 6 three-digit numbers. Among these numbers, it is known that:

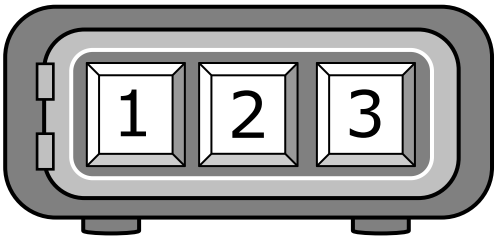
- Only one number is divisible by 16.
- Excluding the above number, only one number is divisible by 4.
- Excluding the above two numbers, only two numbers are divisible by 2.
- One number is a prime number.
- One number is a square of a prime number.

What is the largest three-digit number that can be formed by using digits A, B and C exactly once?

8. In the diagram below, there are 14 points on a circle. In how many ways can we draw 7 non-intersecting chords between these points? (Note that the chords cannot have common ends.)



9. My father's vault is protected with a code that can only be unlocked by entering a three-digit combination password, with each digit being either 1, 2 or 3.



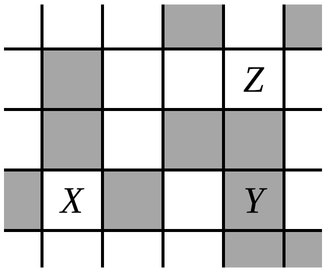
The vault will only be opened if the buttons corresponding to the three digits of the password are pressed in succession. For example, anyone can press 1, 2, 3, 1, 2 and the vault will unlock if the password happens to be 312.
My mother doesn't know the password of the vault, but she wants to open it. What is the minimum number of buttons that she must press to be sure that the vault will be opened? Note that the vault does not have an [ENTER] button.

10. For any positive real number x, we define $[x]$ to be the greatest integer that does not exceed x and $\{x\}$ to be the fractional part of x such that $\{x\} = x - [x]$. Let a, b and c be positive real numbers that satisfy the following system of equations:

$$\begin{cases} a + 2[b] + 2\{c\} = 4.6 \\ [a] + \{b\} - c = 1 \\ 2\{a\} + 2b + [c] = 4.5 \end{cases}$$

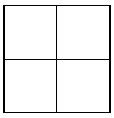
What is the largest possible value of $a + b + c$?

11. In an infinite grid, each unit square is coloured either white or grey. If squares X and Y are located in the same row while squares Y and Z are located in the same column such that squares X and Z are white and square Y is grey, then the ordered triple (X, Y, Z) is called an "*IMC triangle*". For example, as shown in the grid below, (X, Y, Z) is an *IMC triangle*.



What is the maximum possible number of "*IMC triangles*" that can be found on a 15×15 grid?

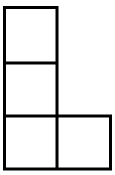
12. Each square of an infinite grid has been filled with exactly one digit. Elsa places an O-tetromino on this grid covering four squares along the grid lines:



The minimum number of different digits used to fill the infinite grid so that the four squares will have different digits, regardless the position of the O-tetromino, is 4. The following grid is an example of such filling.

1	2	1	2	1	2	1	2
3	4	3	4	3	4	3	4
1	2	1	2	1	2	1	2
3	4	3	4	3	4	3	4
1	2	1	2	1	2	1	2
3	4	3	4	3	4	3	4
1	2	1	2	1	2	1	2
3	4	3	4	3	4	3	4

When Elsa places a L-tetromino on this grid covering four squares along the grid lines, what is the minimum number of different digits that must be used for the infinite grid so that the four squares covered by the L-tetromino will always have different digits? (Elsa may flip or rotate the L-tetromino)



L-tetromino

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Team:____________________ ***Name***: ___________________ ***ID.***: _______________

Section B.
Answer the following 3 questions, and show your detailed solution in the space provided after each question. Each question is worth 20 points.

1. Consider the following finite sequence of numbers: $\sqrt{1}$, $\sqrt{2}$, $\sqrt{3}$, …, $\sqrt{2023}$, $\sqrt{2024}$. Now, at each turn, select two numbers, say x and y, from the sequence which are to be removed and then replace them in the sequence by a single number:

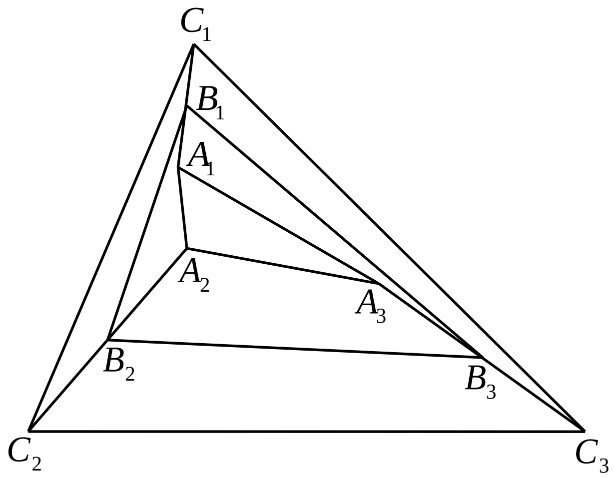
$$z=\sqrt{\frac{1}{\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{x^2y^2}}}.$$

After doing this 2023 times, a single number is left. What is this number?

Answer: ______________________________

Team: ____________________ ***Name:*** ____________________ ***ID.:*** ________________

2. In the diagram below, $A_1A_2A_3$, $B_1B_2B_3$ and $C_1C_2C_3$ are three triangles such that $A_1A_2A_3$ is drawn inside $B_1B_2B_3$ and $B_1B_2B_3$ is drawn inside $C_1C_2C_3$, where B_i is the midpoint of A_iC_i for all $1 \le i \le 3$.



If the perimeters of $A_1A_2A_3$ and $C_1C_2C_3$ are a cm and c cm, respectively, what is the maximum possible value of the perimeter, in cm, of $B_1B_2B_3$?

Answer: ____________________ cm

Team: ____________________ ***Name***: ____________________ ***ID.***: ________________

3. Anne and Bob will play a number game, where they use one piece of paper with only the number 2024 written on it. Starting with Anne, each player takes turns writing down a positive integer that is at least one-third of the previous number but less than the previous number. For the first step, Anne should write a number less than 2024 but no less than $\dfrac{2024}{3}$. The player who writes down the number 1 wins the game. Determine, with complete proof, which player has a winning strategy? What is the strategy?

Answer: ________________ has a winning strategy