2013 Upper Primary Division Second Round Solution

1. What is the value of $57.6\times\frac{8}{5}+28.8\times\frac{184}{5}-14.4\times 80$?

   (A) 0 (B) 8 (C) 14.4 (D) 38.8 (E) 57.6

【Solution】

Note that $57.6=2\times 28.8=4\times 14.4$. Hence the given expression is equal to

$$57.6\times(\frac{8}{5}+\frac{1}{2}\times\frac{184}{5}-\frac{1}{4}\times 80)=57.6\times(\frac{8}{5}+\frac{92}{5}-\frac{100}{5})=57.6\times 0=0.$$

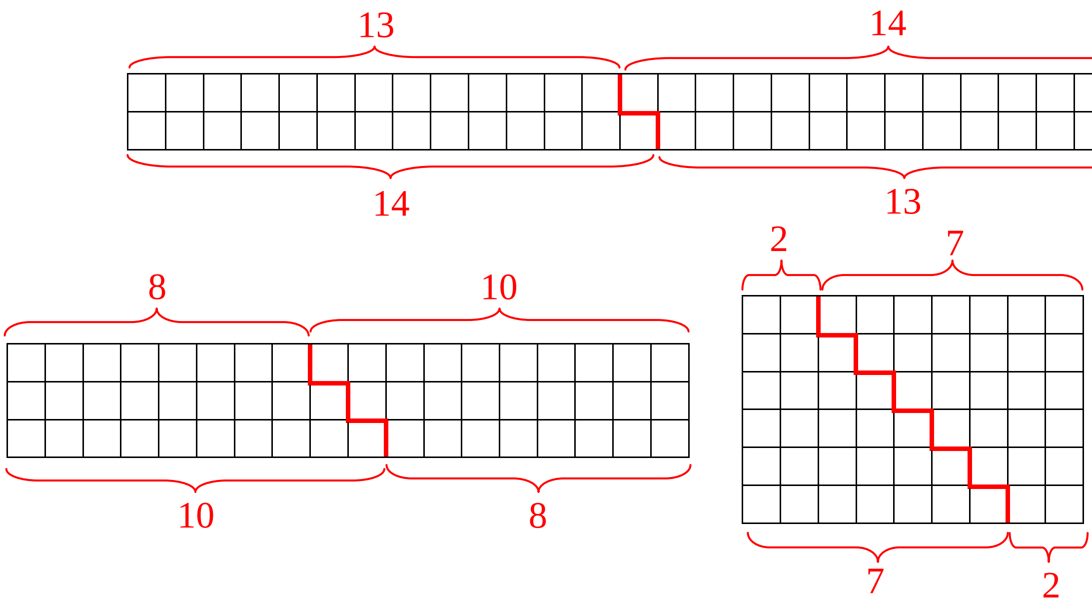
Answer：(A)

2. Max gives 27 apples to a group of friends. The numbers of apples they receive are consecutive positive integers. What is the maximum size of this group?

   (A) 2 (B) 3 (C) 4 (D) 5 (E) 6

【Solution 1】

A sum of consecutive positive integers may be represented geometrically as a staircase. Two identical staircases can be put together to form a rectangle. If the sum of the consecutive integers is 27, the area of the rectangle is 54. Not counting the 1 × 54 rectangle, there are three others with integral dimensions, namely, 2 × 27, 3 × 18 and 6 × 9. They are shown in the diagram, partitioned into two identical staircases. The sums they generated are 13+14, 8+9+10 and 2+3+4+5+6+7. The answer is (E).



【Solution 2】

There are three ways of expressing 27 as a sum of consecutive positive integers, namely, 27 = 13+14, 27 = 9+9+9 = 8+9+10 and 27 = 9+9+9 = (4+5)+(3+6)+(2+7) = 2+3+4+5+6+7. It follows that 27 can be expressed as a sum of at most 6 consecutive positive integers.

Answer：(E)

3. What is the sum of all possible two-digit perfect squares such that the sums of their digits are also perfect squares?

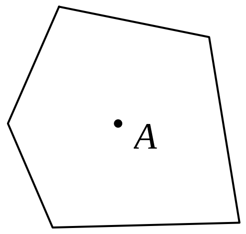
（A） 100 （B） 110 （C） 117 （D） 181 （E） 271

【Solution】

The two-digit perfect squares are 16, 25, 36, 49, 64 and 81. We have 1+6 = 7 = 2+5, 4+9 = 13, 6+4 = 10 and 3+6 = 9 = 8+1. Hence the desired sum is 36+81 = 117.

Answer：(C)

4. The diagram shows a point A inside a pentagon with area 20 cm^2. If the distance from A to each side of the pentagon is 5 cm., what is the perimeter, in cm, of the pentagon?

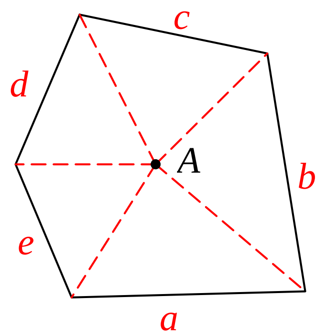


（A） 4 （B） 8 （C） 10

（D） 15 （E） 20

【Solution】

The diagram shows the partition of the pentagon into five triangles whose bases are the sides of the pentagon, with lengths a, b, c, d and e respectively. Since the height of each triangle is 5 cm, their total area, in cm^2, is $20=\frac{1}{2}\times 5\times(a+b+c+d+e)$),



so that $a+b+c+d+e=20\times\frac{2}{5}=8$cm.

Answer：(B)

5. What is the sum of the digits of the number whose square is equal to 15984×48951?

（A） 18 （B） 21 （C） 24 （D） 27 （E） 36

【Solution 1】

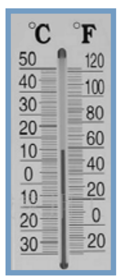
Note that $15984=4\times 4\times 999$ and $48951=7\times 7\times 999$. Hence the desired number is equal to $4\times 7\times 999=27972$ and 2+7+9+7+2=27.

【Solution 2】

Note that $15984=16000-16=16\times(1000-1)$ and $48951=49000-49=49\times(1000-1)$. Hence the desired number is equal to $\sqrt{16\times 49\times(1001-1)}=2800-28=27972$, and 2+7+9+7+2=27.

Answer：(D)

6. The usual thermometer has two scales, Celsius measured in °C and Fahrenheit measured in °F. If m°C is the same temperature as n°F, the conversion formula is $m\times\frac{9}{5}+32=n$. What is the value of m if m°C is the same temperature as n°F and $m+n=60$?

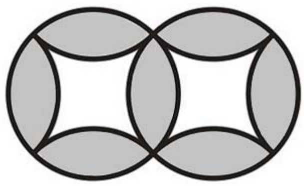


【Solution】

We have $60=m+n=m+m\times\frac{9}{5}+32$. Hence $28=m\times\frac{14}{5}$ so that $m=10$.

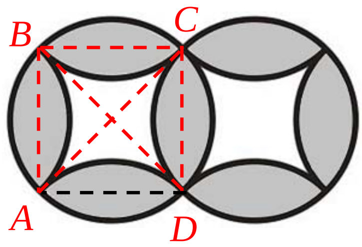
Answer：10°C

7. The diagram shows two intersecting circles of radius 10 cm. The four arcs inside each circle have the same shape and equal length. Taking $\pi \approx 3.14$, what is the area, in cm^2, of the shaded region which consists of 7 identical parts?



【Solution】

Connect *AB*, *BC*, *CD*, *DA*, *AC* and *BD* as shown in the diagram. Then *AC* = *BD* = 20 cm and the area of *ABCD* is $\frac{1}{2} \times AC \times BD = 200\,\text{cm}^2$. The area of each of the intersecting circles is $\pi \times 10^2 = 314\,\text{cm}^2$. Let *S* cm^2 be the area of each of the 7 identical parts of the shaded region. By symmetry, 2*S* cm^2 is the difference of the area of the circle and the square *ABCD*. Hence $2S = 314 - 200 = 114$ so that *S* = 57. It follows that the desired area is 7*S* = 399 cm^2.



Answer：399 cm^2

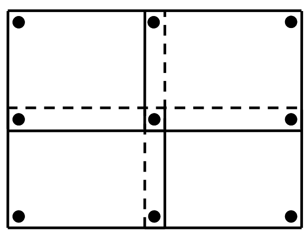
8. The total number of players on three badminton teams is 29. No two players on the same team play against each other, while every two players on different teams play each other exactly once. What is the maximum number of games played?

【Solution】

Suppose team A has at least two more players than team B. Transfer one player X from team A to team B. Before the transfer, X plays every player on team B. After the transfer, X plays every player left on team A, which is still more than the number of players originally in team B. So the total number of games played has increased. It follows that to maximize the total number of games played, the size of the teams should not differ by more than 1. Hence we should have 10 players in two of the teams and 9 players in the third team. The total number of games is then $10 \times 10 + 10 \times 9 + 10 \times 9 = 280$.

Answer：280 games

9. A class is putting up 40 rectangular posters of the same shape and size on a wall. Each poster must be held in place by one nail near each corner. Adjacent posters may overlap slightly so that the same nail can serve to hold both of them. The diagram shows how 9 nails can hold four posters adjacent diagonally. What is the minimum number of nails required to hold all 40 posters?



【Solution 1】

Clearly, the 40 posters should be arranged in a rectangular array, and there are four ways to do so.

For a 1×40 array, the number of nails required is $(1+1) \times (40+1) = 82$.

For a 2×20 array, the number of nails required is $(2+1) \times (20+1) = 63$.

For a 4×10 array, the number of nails required is $(4+1) \times (10+1) = 55$.

For a 5×8 array, the number of nails required is $(5+1) \times (8+1) = 54$.

Hence the minimum number of nails required is 54.

【Solution 2】

Clearly, the 40 posters should be arranged in a rectangular array, and there are four ways to do so. We wish to maximize the number of four-way common corners.

For a 1×40 array, there are $(1-1)\times(40-1)=0$ such corners.

For a 2×20 array, there are $(2-1)\times(20-1)=19$ such corners.

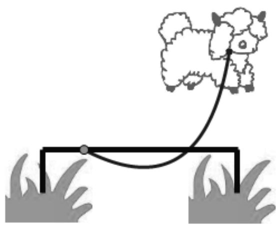
For a 4×10 array, there are $(4-1)\times(10-1)=27$ such corners.

For a 5×8 array, there are $(5-1)\times(8-1)=28$ such corners.

Hence the minimum number of nails required is $28+2\times(5+8)=54$.

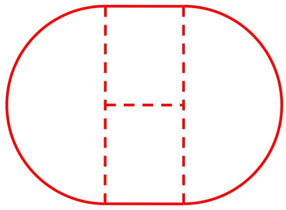
Answer：54 nails

10. The diagram shows a sheep on a lawn, tied by a string to a metal rod parallel to the ground. The length of the string is 10 m, and the length of the metal rod is 3 m. The sheep is of negligible size and the height of the rod above the ground is also negligible. Taking $\pi\approx 3.14$, what is the area, in m^2, of the part of the lawn from which the sheep can eat the grass?



【Solution】

The part of the lawn from which the sheep can eat the grass is shown in the diagram, consisting of two semicircle of radius 10 m and a rectangle of width 3 m and height 20 m, where the line in the middle of the rectangle represent the rod. The area, in m^2, is $\pi\times 10^2+3\times 20=374$.



Answer：374 m^2

11. Leon uses a code to convert a letter string consisting only of As, Bs and Cs, into a number string consisting only of 0s and 1s, by replacing A with 101, B with 11 and C with 0. If the number string obtained is 1110101011111100110101, what is the number of letters in the original letter string?

【Solution】

If the first number of a string is 0, it must stand for C. If it is a 1, and the next letter is another 1, the two 1s must stand for B, If the next letter is 0, then the third letter must be 1 and these three letters must stand for A. Thus the number string can be read in a unique way. In particular, 11-101-0-101-11-11-0-0-11-0-101 must stand for BACABBCCBCA, and the number of letters is 11.

Answer：11

12. Divide the ten positive integers from 1 to 10 into two groups so that when the product of the numbers in the first group is divisible by the product of the numbers in the second group, the quotient is a positive integer. What is the minimum value of this quotient?

【Solution】

The number 7 does not divide any of the other numbers. So it must be placed in the first group. Since it is not divisible by any of the other numbers apart from 1, the quotient is no smaller than 7. The product of the remaining nine numbers is

$2^8\times 3^4\times 5^2$. If we can make the product of the numbers in the second group equal to $2^4\times 3^2\times 5^1$, then the quotient will attain its minimum value of 7. This is indeed possible, as we can put 1, 2, 4, 9 and 10 in the second group, while adding 3, 5, 6 and 8 to 7 in the first. It follows that the minimum quotient is $\frac{3\times 5\times 6\times 7\times 8}{1\times 2\times 4\times 9\times 10}=7$.

Answer：7

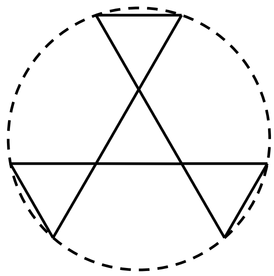
13. Wally writes down three positive integers *a*, *b* and *c* in a row on the blackboard, where *a* + *c* = 2*b*. He then erases the commas between the three numbers to obtain a five-digit number. What is the maximum value of this number?

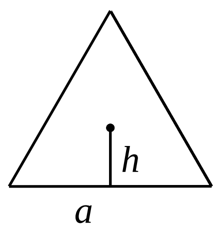
【Solution】

None of *a*, *b* and *c* can be a three-digit number and *b* must be a two-digit number. To maximize the five-digit number, one of *a* and *c* must be 9 and the other must be 99, so that $b=\frac{9+99}{2}=54$. Comparing 95499 with 99549, we see that the maximum value is 99549.

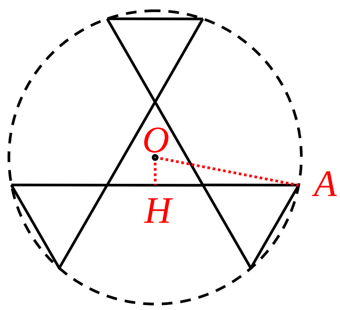
Answer：99549

14. In an equilateral triangle of side length *a*, the distance *h* from the centre of the triangle to any side satisfies $a^2=12h^2$. Lily uses four equilateral triangles of side length 6 cm to make a cardboard windmill with three blades. Two triangles sharing a common vertex have the corresponding sides lying on the same straight line, as shown in the diagram. Taking $\pi\approx 3.14$, what is the area, in cm², of the circle swept out by the blades of the windmill?





【Solution】



Let *O* be the centre of the windmill, *A* be a corner of one of the blades, and *H* is such that *OH* is perpendicular to *AH*.

Now $AH=6+\frac{6}{2}=9\,\text{cm}$ while $OH^2=\frac{6^2}{12}=3\,\text{cm}^2$.**(5 marks)**

By Pythagoras' Theorem,

$$OA^2=OH^2+AH^2=3+81=84\,\text{cm}^2. \quad \textbf{(10 marks)}$$

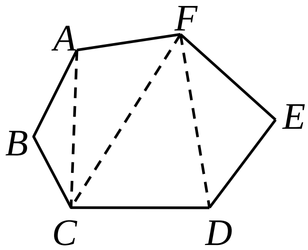
Hence the area of the circle is $\pi\times 84=263.76\,\text{cm}^2$. **(5 marks)**

Answer：263.76 cm²

【Marking Scheme】

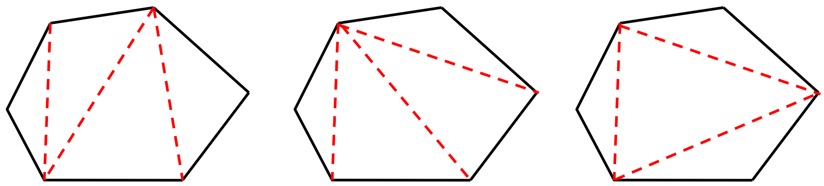
Find OH^2（i.e. $h^2=3$）, 5 marks.
Using Pythagoras' Theorem to find OA^2=84, 10 marks.
Using formula correctly to find the area of the circle, 5 marks.
Only correct answer without solving process, 5 marks.

15. The diagram shows a hexagon *ABCDEF* partitioned into four triangles by three diagonals *AC*, *CF* and *FD*, no two of which intersect except at the vertices. What the total number of ways of partitioning *ABCDEF* into four triangles with three non-crossing diagonals?



【Solution】

There are only three such configurations of three non-crossing diagonals, as shown in the diagram. **(5 marks)**



In the first, the three diagonals form a broken polygonal line. The starting point can be any of the six vertices. **(5 marks)** In the second, the three diagonals all converge on a single vertex, and this can be any of the six. **(5 marks)** In the third, the three diagonals form a triangle joining alternate vertices. **(5 marks)** This yields only two ways. Hence the total number of ways is 6+6+2=14.

Answer：14 ways

【Marking Scheme】

Classify all situations correct, 5 marks.
Each correct number of cases of each situation, 5 marks.
Only correct answer without solving process, 5 marks.